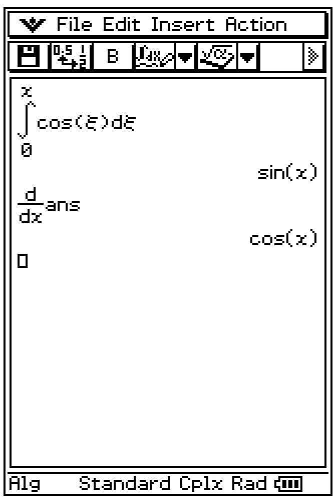
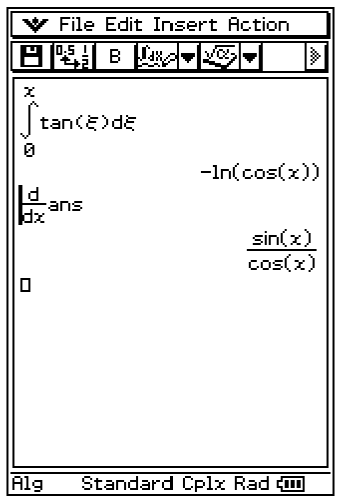
Fundamental Theorem of Calculus

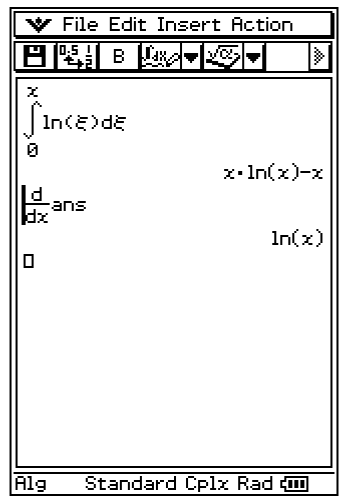
For the functions $f(x) = x^3$ and $f(x) = x^2$ we saw that the derivative of the integral of f was in fact f. That is:

$$\frac{d}{dx}\int_0^x f(\xi)d\xi = f(x)$$

Is this generally true? Try a few examples in ClassPad to start convincing yourself.



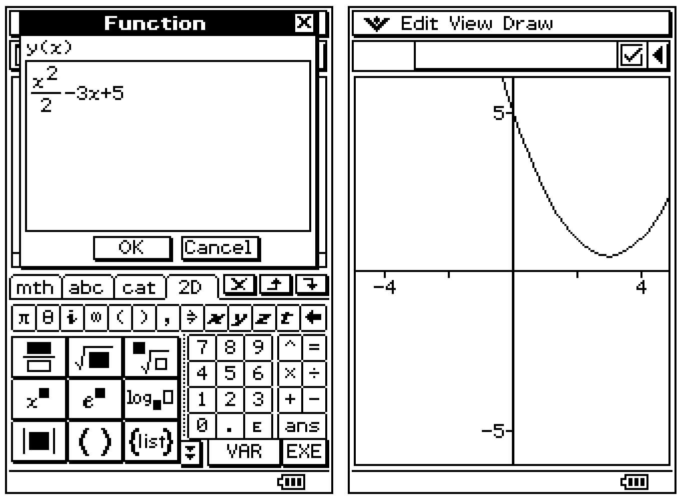




This is a very important observation. So important it is given the name of the Fundamental Theorem of Calculus. We'd better examine it a little.

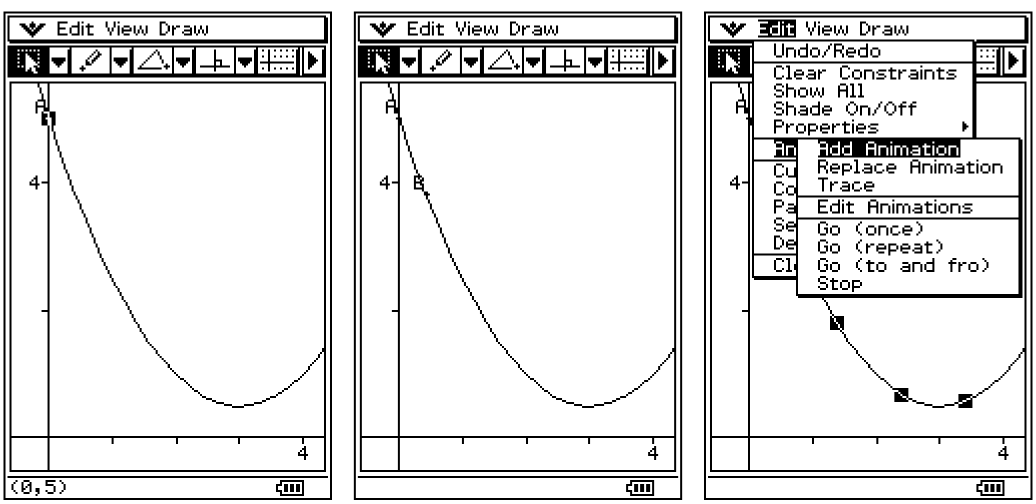
Graphical Analysis

Let's first look at the theorem in terms of graphs in a geometry window. Create the following graph:

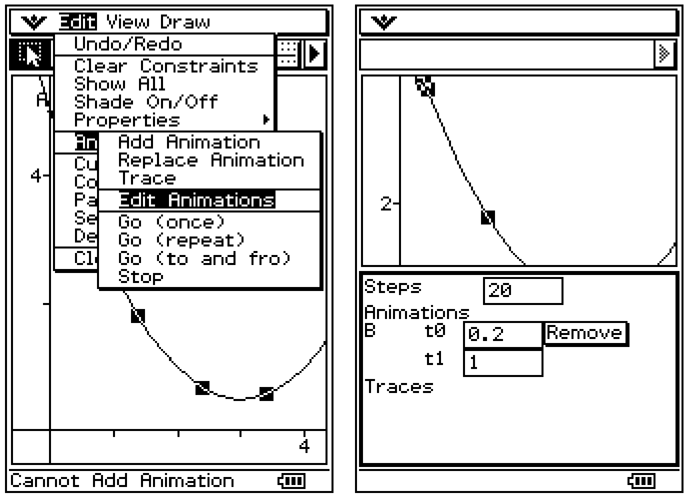


We'll use the Animation capabilities of ClassPad to examine the trapezoidal approximation to the integral.

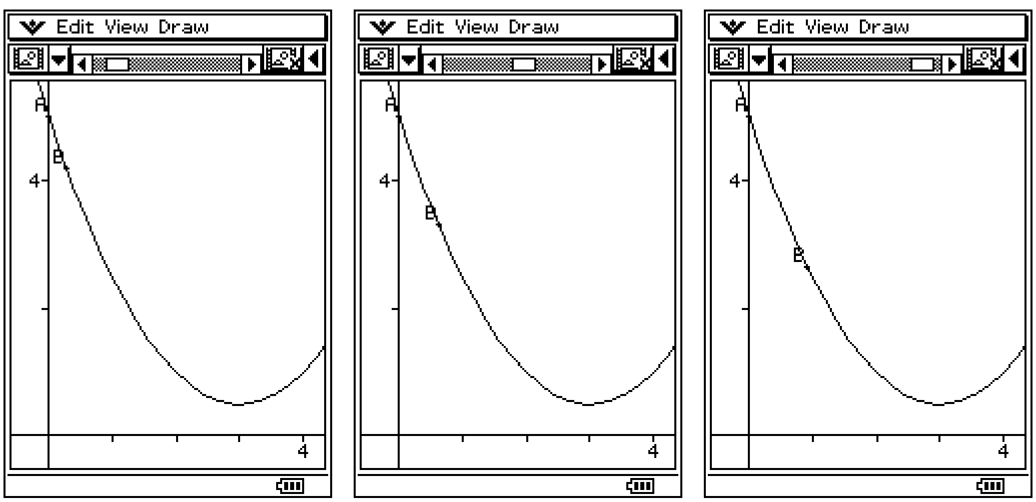
First we'll create a point A on the curve at x=0. Then we create a second point B on the curve, select B and the curve and Edit/Animate/Add Animation:



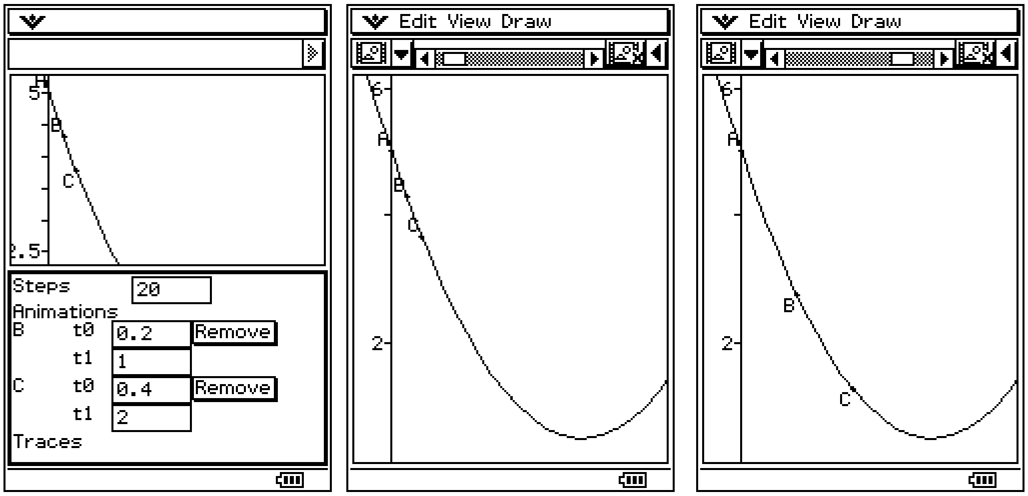
Now, we'll start B at the point with x=0.2 and end it at the point with x=1. To do this we go int the Edit Animations window:



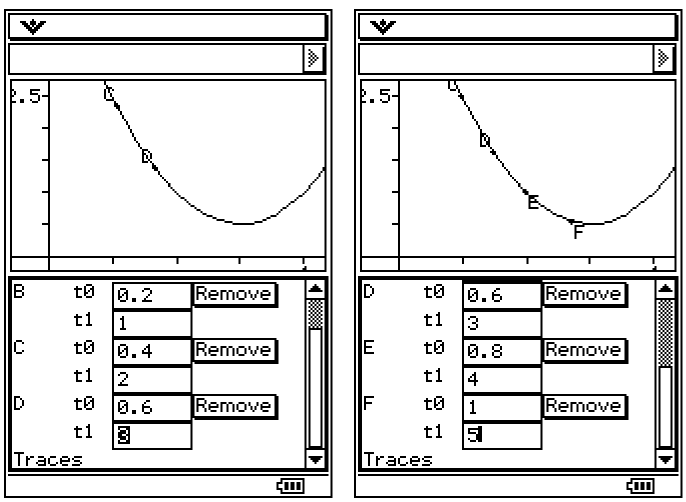
Try running the animation: you'll see B move along the curve as specified.



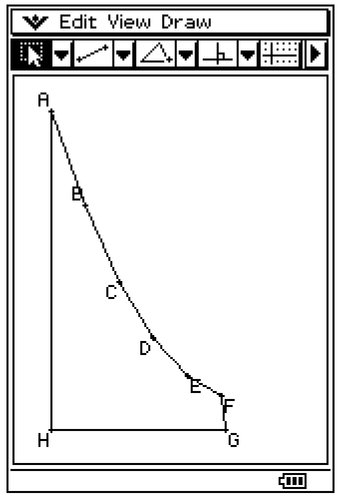
We now want a third point C, which we will run from 0.4 to 1.2:



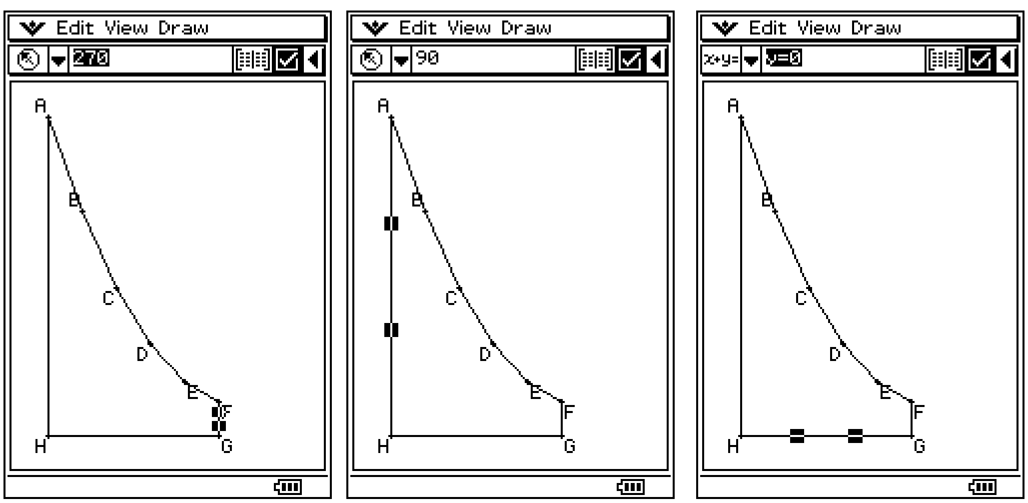
We now add 3 more points D, E and F, setting their animations appropriately:



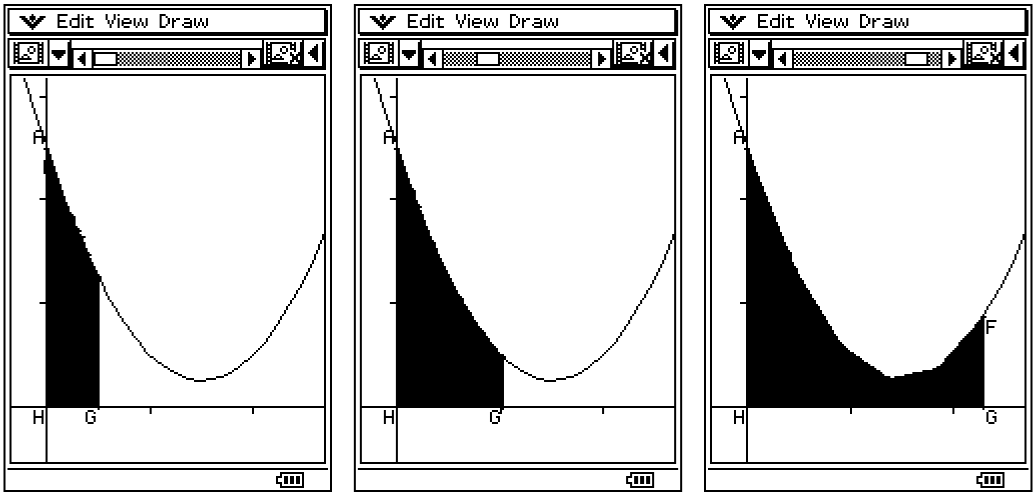
Our next step is to join up the dots and construct the polygon whose are approximates the area under the curve. Hide the curve and the axes to make it easier to see:



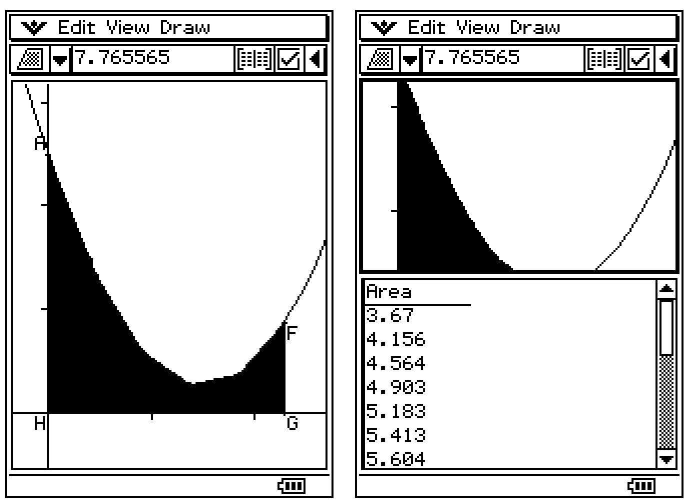
We need to make sure that the line FG is vertical. We can do this by setting its direction. While we are at it we can set the direction of AH and the equation of GH to make sure they remain the axes:



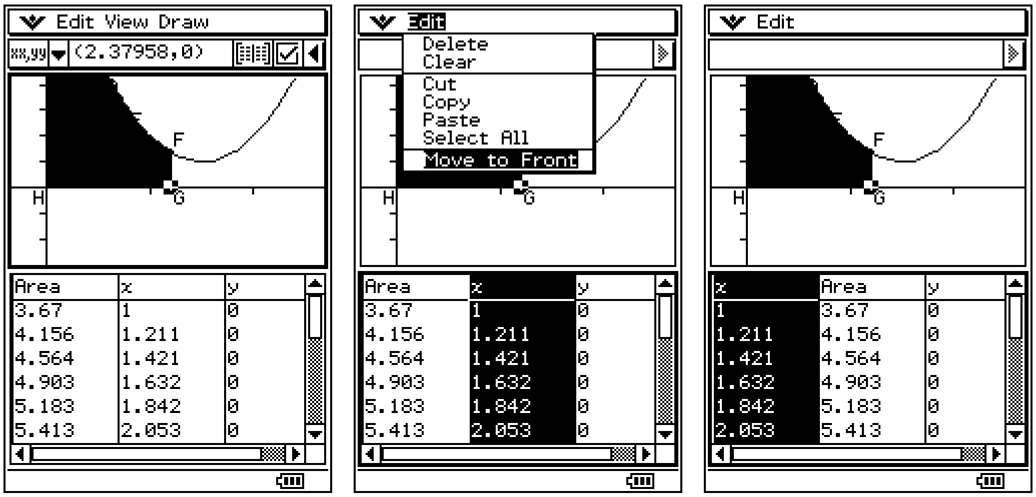
We can now shade this polygon and animate:



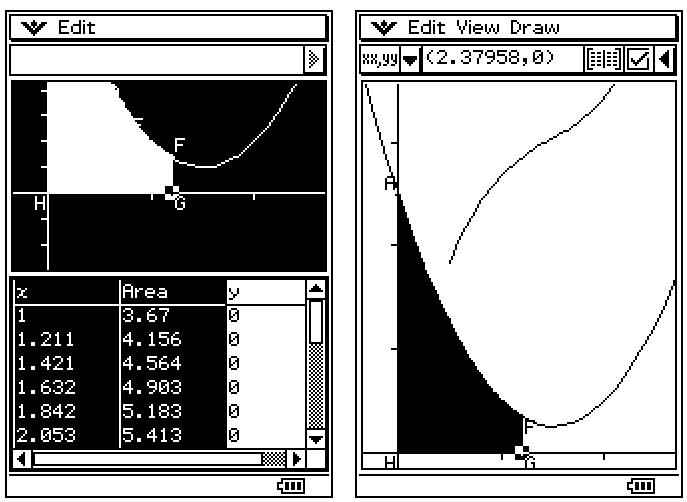
We can look at the shaded area simply by leaving nothing selected and bringing up the measure box (by default the shaded area is displayed). Pressing the Table button will tabulate the area values:



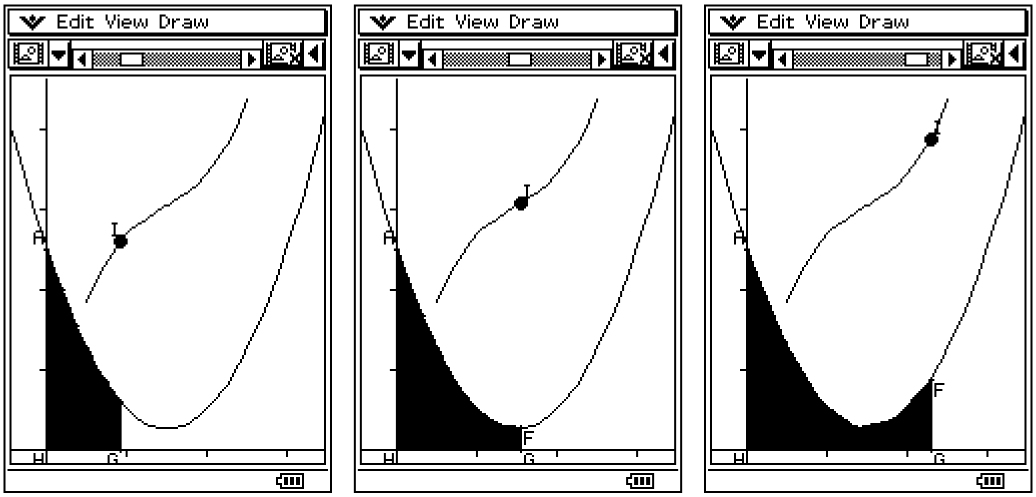
Tabulate the location of point G by selecting it and pressing the Table button. You can then use Edit/Move to Front to put the x coordinates in front of the Area.



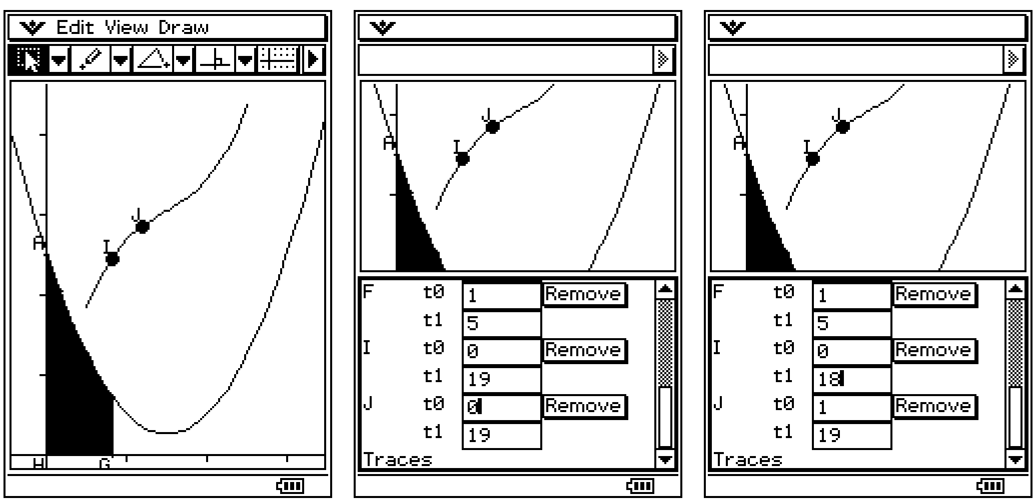
Dragging both columns of the table back into Geometry gives a graph of the area of the polygon:



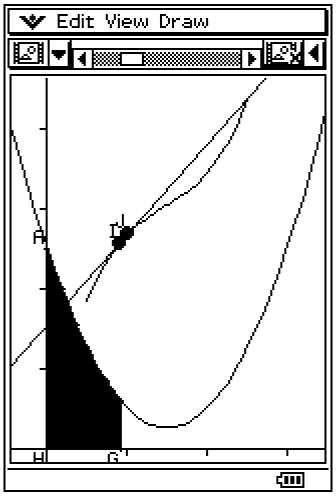
Next step is to run a point along the new curve. Create the point I on the curve, and then add an animation. We see that I runs along the area curve as our polygon expands:

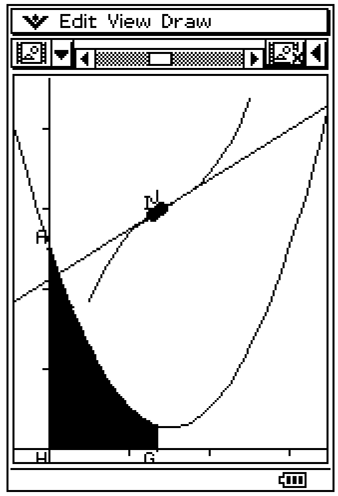


Next step is to add another point J to run along the area curve. Initially, the animations of I and J are both set to run along the curve from parameter value 0 to parameter value 19 (these parameter values refer to the 20 points captured in our table.) We reset the end values so that I runs from 0 to 18 and J runs from 1 to 19:

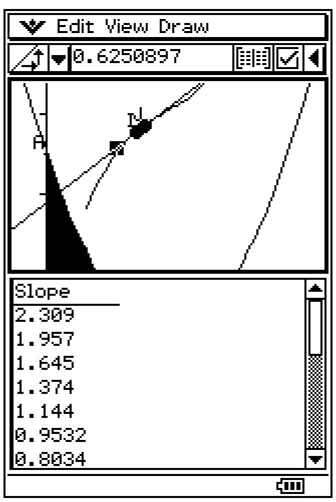


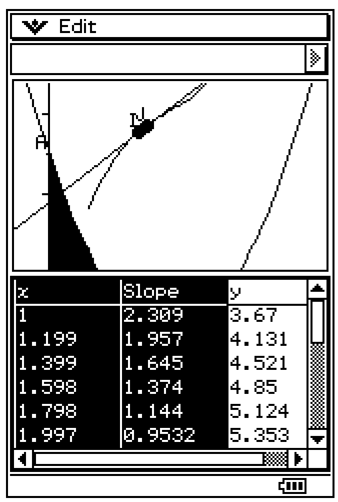
Create a line through I and J. This is a chord to the curve.

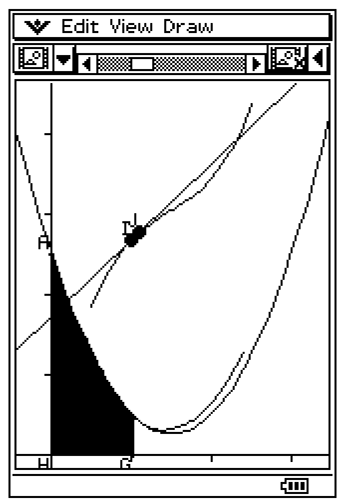




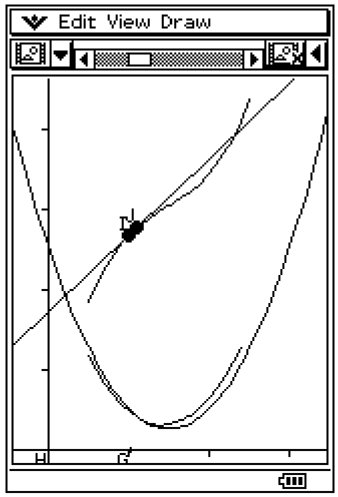
Finally, tabulate the slope of the chord against the x coordinate of I:





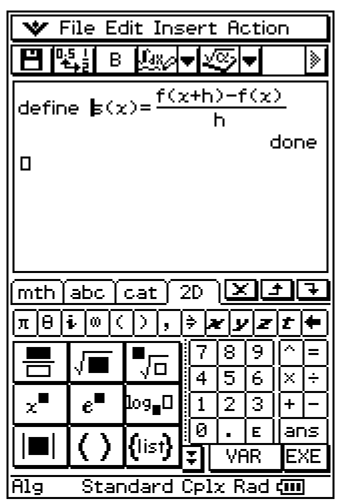


Hiding the polygon away, we can observe that the resultant curve is pretty close to the original.



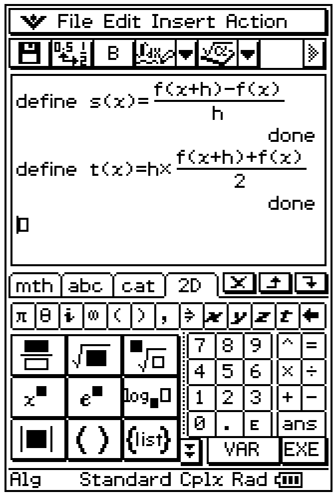
Algebraic Analysis

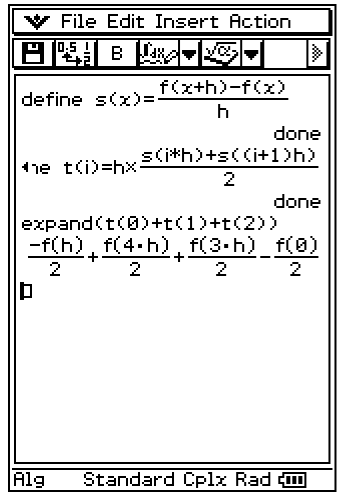
Let's examine the trapezoidal area / chord slope relationship in a purely algebraic context. Given a function f(x), let s(x) be the slope of the chord starting at x and ending at x+h:



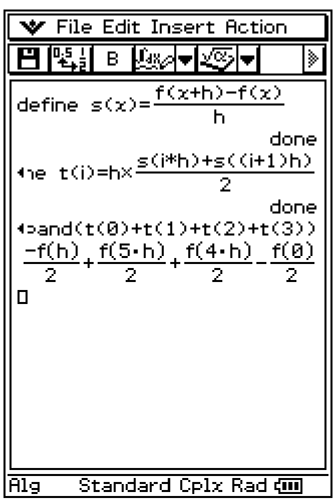
Now let's approximate the area under the curve s(x) using trapezoids of width h.

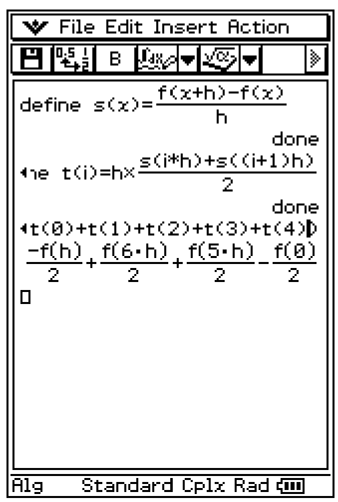
We write t(i) for the i^{th} trapezoid and add up 3 of them to start with:

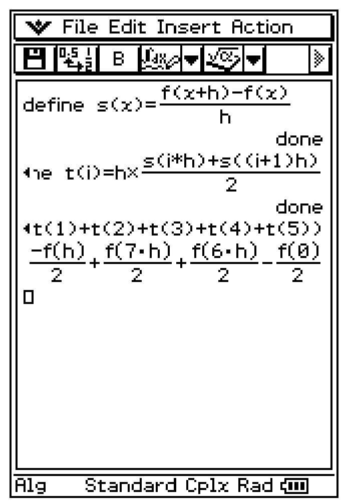




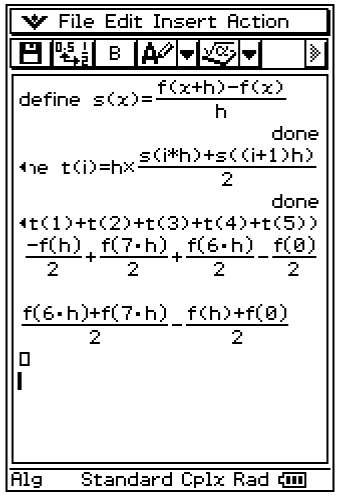
We notice that the formula is not quite as complicated as it could be as it has no term which involves f(2.h). Big deal, we got 4 terms instead of 5. But the saving is more pronounced if we split the problem up into 4 trapezoids, or 5, or 6 (assuming that as we add trapezoids, we decrease the size of h so that with n trapezoids, n.h = x)







All the middle terms disappear, and we notice we can rearrange the terms as follows:



As 6h=x, this can be rewritten:

$$\frac{f(x)+f(x+h)}{2}-\frac{f(0)+f(h)}{2}$$

The above analysis was all algebraic, and nowhere involved limits. The important point was that, because of the close relationship between the formula for the slope of a chord and for the area of a trapezium of the same width, most of the terms in the trapezoidal approximation for the area cancelled out.

It is suggestive to note that as h becomes small, we can expect

$$f(x+h)\to f(x)$$

and

$$f(h)\to f(0)$$

hence

$$\frac{f(x)+f(x+h)}{2}-\frac{f(0)+f(h)}{2}\to f(x)-f(0)$$